Mathematics Written Exam

- **Date:** May 27, 2020
- **Duration:** 2 hours
- **Instructions:**
  - This exam has two groups: I and II.
  - Group I has seven multiple-choice questions and four possible answers to each one, of which only one is correct. In each question, circle the answer you think is correct, not showing the calculations made. If you circle more than one answer for the same question, the question will be considered unanswered. Each right, wrong and blank or invalid answer is worth, respectively, 1, $-\frac{1}{3}$ and 0 points. The minimum total number of points of this group is 0.
  - Group II has three open-ended questions, the first with four parts, the second with six parts and the third with 1 part. The grade of each question is written before its text. Show all the calculations and justify all the reasonings made. If you need to round up numbers in intermediate steps, use two decimal places. Answer each question in the correspondent space and use the front and back of each sheet.
  - No questions will be answered during the exam. If you need to assume something while answering a question, state it, and be consistent with what you assumed in the steps that follow.
  - Only use writing material and a calculator. Do not use cell phones or other material.
  - Do not unstaple this exam.
  - The last two pages of this exam have formulae and space for drafts, whose contents will not be graded.

Name:

Group I

1 Let Ω be the result space associated to a random experiment and let A, B and C be three events $(A \subseteq \Omega, B \subseteq \Omega \text{ e } C \subseteq \Omega)$. It is known that $P(A) = P(B)$, $P(A \cap B) = 0$ and $P(C \mid A) = P(C \mid B)$. Which of the following statements may be false?

a) $P(A \cap C) = P(B \cap C)$

b) $P(A \cap B \cap C) = 0$

c) $P(A \cup B \cup C) = 1$

d) $P(A \mid C) = P(B \mid C)$

2 The element of each position of the 5^{th} row of Pascal's triangle was written in a ball indistinguishable by touch from the others, and the balls were placed in a bag (which means that the bag has balls with the same number). Then, two balls were randomly withdrawn from the bag. What is the probability that the sum of numbers written on the withdrawn balls was even?

a) $\frac{3}{10}$

b) $\frac{2}{5}$

c) $\frac{3}{5}$

d) $\frac{1}{2}$

3 Consider the functions $f: \mathbb{R} \to \mathbb{R}$, continuous, even, strictly increasing in $\mathbb{R}^+$ and such that $f(-4) = 1$ and $f(2) = 0$, and $g: \mathbb{R} \to \mathbb{R}$, defined by $g(x) = \frac{\sqrt{f(x)}}{\ln(f(x))}$. What is the domain of g?

a) $\mathbb{R} \setminus [-2,2]$

b) $\mathbb{R} \setminus \{-4 - 2,2,4\}$

c) $\mathbb{R} \setminus \{-4,4\}$

d) $\mathbb{R} \setminus ([-2,2] \cup \{-4,4\})$

4 Consider a function $f: \mathbb{R} \to \mathbb{R}$, whose graph has as asymptotes the straight lines with equations $y = -1$ and $y = x$. Which of the following pairs may be the pair of equations of the asymptotes of the graph of $|f|$?

a) $y = 1$ and $y = x$

b) $y = -1$ and $y = x$

c) $y = -x$ and $y = x$

d) $y = -1$ and $y = 1$

5 What is the maximum area of the rectangles whose perimeter is 12?

a) 3

b) 12

c) 9

d) 6

6 Consider a function $f: \mathbb{R} \to \mathbb{R}$, such that $f': \mathbb{R} \to \mathbb{R}$ is continuous and such that $f'(0) = -2$, and $f'': \mathbb{R} \to \mathbb{R}$ is positive in $\mathbb{R}^+$ and odd. Which of the following statements is true?

a) f has two maximizers.

b) f has one minimizer and one maximizer.

c) f has two minimizers.

d) f has no minimizers and no maximizers.

7 Consider $n \in \mathbb{N}$. In $\mathbb{C}$, what is the number of non-real solutions of $x^n = 1$?

a) $n - 1$

b) $n - 2$

c) $n - 1$ if n is odd, and $n - 2$ if n is even

d) $n - 2$ if n is odd, and $n - 1$ if n is even

Group II

1 **(4)** A course of Nova School of Business and Economics is having an online exam for all its students. The exam consists in a group with four multiple choice questions, E_1, E_2, E_3 and E_4, a group with three open-ended questions, R_1, R_2 and R_3, and a group with a proof, D. Two versions, a and b, were made for each of the eight questions, and they will be randomly assigned to each exam. The exam of every student will present the multiple-choice group in first place, followed by the open-ended group and, finally, by the proof group. However, the order in which the questions inside each group are presented is randomly chosen. Each multiple-choice question has four answers, but only one of them is right. In a multiple-choice question, each right, not answered and wrong question is worth, respectively, 3, 0 and $p \in \mathbb{R}$ points, and the grade of a student who randomly chooses an answer is 0. The minimum grade of the multiple-choice group is 0 points.

a) **(0,5)** Compute p.

b) **(1)** What is the probability that a student which answers the four multiple choice questions randomly have a grade of 0 points in the multiple-choice group? Show the result rounded to two decimal places.

c) **(1,5)** João and António are taking the exam. What is the probability that they have exactly the same exam, with the same version for each question, and the questions in the same order? Present the result as a simplified fraction.

d) **(1)** Consider the events A: "All question in João's exam show version a" and B: "The questions in the multiple choice of João's exam are presented by increasing order of their indexes, and the same happens with the open-ended group". State and justify whether A and B are independent.

Answer Question 1

2 **(7)** Consider a function $f: \mathbb{R} \to \mathbb{R}$ such that:

f is continuous	$f(5) = 0$
f is strictly decreasing	$\lim_{x \to -\infty} f(x) = +\infty$
$f(3) = 1$	$\lim_{x \to +\infty} f(x) = -2$

Also consider the functions $g: \mathbb{R} \to \mathbb{R}$ and $h: \mathbb{R} \to \mathbb{R}$, defined, respectively, by:

$$g(x) = \begin{cases} \dfrac{1}{f(x)} & \text{if} \quad x < 3 \\ f(x) & \text{if} \quad 3 \leq x \leq 5 \\ -f(x) & \text{if} \quad x > 5 \end{cases} \qquad h(x) = \begin{cases} -\ln(1-x) & \text{if} \quad x < 0 \\ x \cdot e^{\frac{x^2}{4} - \frac{3}{2}x} & \text{if} \quad x \geq 0 \end{cases}$$

a) **(6)** For g and h, do the following:

- **(i)** **(1)** Find their sets of zeros.
- **(ii)** **(1)** Find the limits of their general expressions when x goes to $-\infty$ and $+\infty$.
- **(iii)** **(1,5)** Show that they are continuous in $\mathbb{R}$.
- **(iv)** **(1,5)** Study them regarding monotonicity and the existence of extrema.
- **(v)** **(1)** Find their ranges.

b) **(1)** Show that $g - h$ has at least one zero in $[0,5]$.

Answer Question 2

3 **(2)** In $\mathbb{C}$, the set of complex numbers, consider $a = -2 + 2i$ and $b = 4\,\text{cis}\left(\frac{4}{3}\pi\right)$. Compute the area of the following region (remember that $\arg(z)$, $\overline{z}$, $|z|$ and $\text{re}(z)$ represent, respectively, the argument, the conjugate, the modulus and the real part of z):

$$R = \{z \in \mathbb{C}: \arg(a) \leq \arg(z) \leq \arg(\overline{z}) \wedge |z| \leq |a^2| \wedge \text{re}(z) \leq \text{re}(b)\}$$

Answer Question 3

Formulae

Special Limits

$$\lim\left(\left(1+\frac{1}{n}\right)^n\right) = 1 \qquad (n \in \mathbb{N})$$
$$\lim_{a\to 0}\left(\frac{\sin(a)}{a}\right) = 1$$
$$\lim_{a\to 0}\left(\frac{e^a-1}{a}\right) = 1$$
$$\lim_{a\to 0}\left(\frac{\ln(a+1)}{a}\right) = 1$$
$$\lim_{a\to +\infty}\left(\frac{\ln(a)}{a}\right) = 0$$
$$\lim_{a\to +\infty}\left(\frac{b^a}{a^p}\right) = +\infty \qquad (b > 1, p \in \mathbb{R})$$

Derivation Rules

$$(a+b)' = a' + b'$$
$$(ab)' = a'b + ab'$$
$$\left(\frac{a}{b}\right)' = \frac{a'b - ab'}{b^2}$$
$$(a^p)' = pa^{p-1}a' \qquad (p \in \mathbb{R})$$
$$(p^a)' = \ln(p)\, p^a a' \qquad (p \in \mathbb{R}^+ \setminus \{1\})$$
$$\left(\log_p(a)\right)' = \frac{a'}{\ln(p)\, a} \qquad (p \in \mathbb{R}^+ \setminus \{1\})$$
$$(\sin(a))' = a' \cos(a)$$
$$(\cos(a))' = -a' \sin(a) \qquad (b > 1, p \in \mathbb{R})$$

Complex Numbers

$$\left(\rho\operatorname{cis}(\theta)\right)^n = \rho^n \operatorname{cis}(n\theta) \qquad (n \in \mathbb{N})$$
$$\sqrt[n]{\rho\operatorname{cis}(\theta)} = \sqrt[n]{\rho}\operatorname{cis}\left(\frac{\theta + 2k\pi}{n}\right) \qquad (n \in \mathbb{N}, k \in \{0, \ldots, n-1\})$$

Drafts