NOVA SCHOOL OF BUSINESS & ECONOMICS

Mathematics Written Exam

Date	Duration + Tolerance	Elements of consultation allowed
28 May 2024	2 hours + 15 minutes	Calculator

Instructions

This exam is composed of two compulsory groups.

Group I is composed of multiple choice questions and Group II is composed of open-ended questions.

For each multiple choice question, four answer options are presented, of which exactly one is correct. Indicate your answer by selecting the letter corresponding to the option that you consider to be correct. Each correct, incorrect or blank/null answer in Group I has the classification of, respectively, 1/20, −0.3/20 and 0/20 points. In case more than one option is selected for the same question, the answer will be considered null. The minimum grade in Group I is 0.

All open-ended questions should be carefully justified, and all computations should be presented. The classification for each question in this group is indicated next to it.

Do not unstaple this booklet.

Do not use any type of corrector. If necessary, cross out.

Name of the Candidate

Group I

1 (1/20) Let Ω be the sample space of some random experiment, and let A and B be two events in Ω. Suppose that:

- A and B are independent events
- $P(\bar{A}) = \frac{7}{10}$
- $P(A \cup B) = \frac{3}{4}$

What is the value of $P(B)$?

a) 5/20

b) 11/20

c) 9/14

d) None of the previous options.

2 (1/20) Rita and Francisco invited three friends to go with them to the cinema. They bought five tickets, with consecutive numbers for some given row, and distributed them randomly. What is the probability of Rita and Francisco sitting next to each other?

a) 2/5

b) 3/5

c) 4/5

d) None of the previous options.

3 (1/20) Consider the sequence (u_n) with general term

$$u_n = \begin{cases} \dfrac{\sqrt{n}\,\operatorname{sen}(e^n\pi)}{n\sqrt{2^n+3}} + \ln\left(\left(1 - \dfrac{1}{n+2}\right)^n\right), & \text{if } n \text{ odd} \\ \left(\sqrt{e^{-\pi n}} - 1\right) + (-1)^{3n}.\dfrac{4^n + 2^{n+1}}{5^n + 2^6}, & \text{if } n \text{ even} \end{cases}$$

Which of the following propositions is true?

a) $\lim_n u_n = 0$

b) $\lim_n u_n = -1$

c) The sequence (u_n) does not admit limit.

d) None of the previous options.

4 (1/20) Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = |x^3 + x|,$$

for all $x \in \mathbb{R}$. Which of the following propositions is true?

a) f is an even and injective function.

b) f is an even function and, therefore, not injective.

c) f is an odd function, but not injective.

d) None of the previous options.

5 (1/20) What is the value of the following limit?

$$\lim_{x \to 2} \frac{x^3 - 8x^2 + 21x - 18}{x - 2}$$

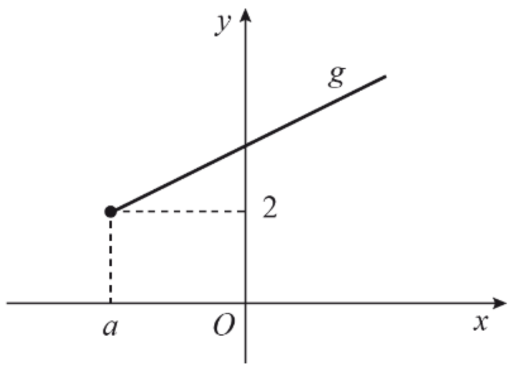
a) 1

b) -1

c) 2

d) None of the previous options.

6 (1/20) The figure below shows, on an orthonormal coordinate system xOy, part of the graph of a function g, with domain $[a, +\infty[$, for some $a < -\frac{1}{3}$.



Suppose that the function $f: \left]-\infty, -\frac{1}{3}\right[\to \mathbb{R}$, defined by

$$f(x) = \begin{cases} \log_3\left(-x - \frac{1}{3}\right), & \text{if } x < a \\ g(x), & \text{if } x \geq a \end{cases},$$

is continuous on $\mathbb{R}$. What is the value of a?

a) $-19/3$

b) $-25/3$

c) $-28/3$

d) None of the previous options.

7 (1/20) Let f be the function with domain $\mathbb{R}$ defined by $f(x) = e^x - 3$. On which of the following intervals does Bolzano's Theorem apply to establish that the equation $f(x) = -x - \frac{3}{2}$ has, at least, one solution?

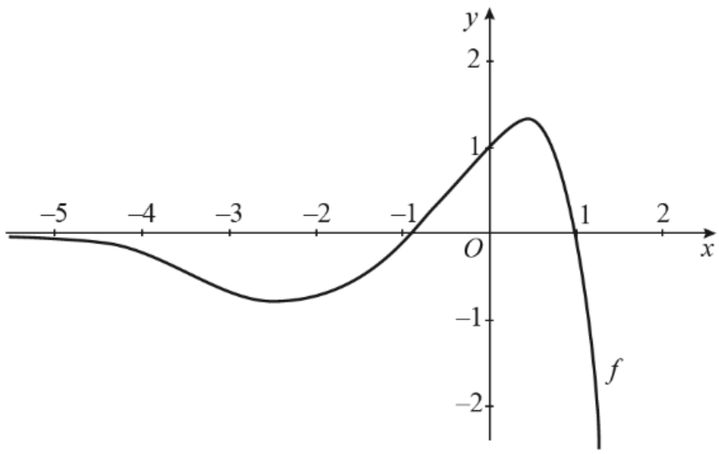
a) $\left]0, \frac{1}{2}\right[$

b) $\left]\frac{1}{2}, 1\right[$

c) $\left]1, \frac{3}{2}\right[$

d) None of the previous options.

8 (1/20) The figure below shows, on an orthonormal coordinate system xOy, part of the graph of a function f with domain $\mathbb{R}$.



Let f' and f'' be the first and second order derivatives of f, respectively. Which of the following values can be positive?

a) $f'(-3)$

b) $f''(-3) \times f''(0)$

c) $f(0) \times f'(0) \times f''(0)$

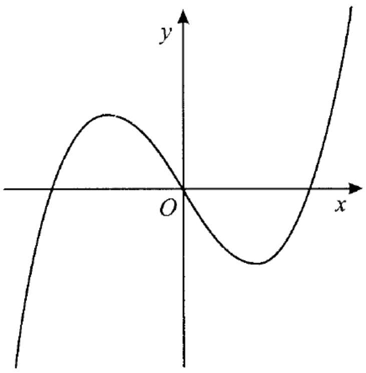
d) None of the previous options.

9 (1/20) Let $f: \mathbb{R} \to \mathbb{R}$ be a function with **second order derivative** given by

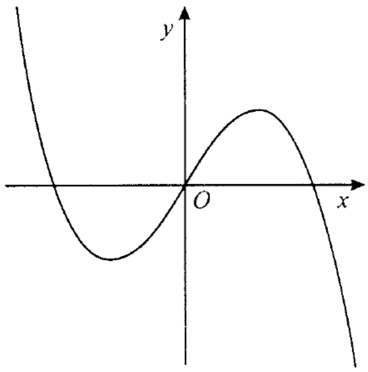
$$f''(x) = 1 - x^2,$$

for all $x \in \mathbb{R}$. In which of the following options can part of the graph of **function f** be found?

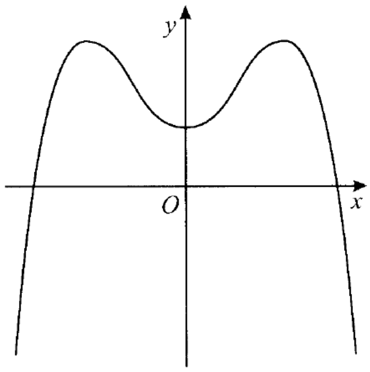
(A)



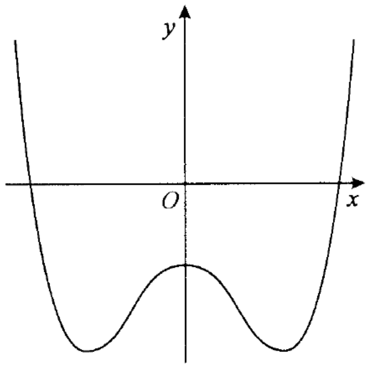
(B)



(C)



(D)



10 (1/20) If $xy = 4$ and $x - y = -2$, what is the value of

$$\frac{x^3+y^3}{x^2-y^2} - \frac{x^2+y^2}{x-y}?$$

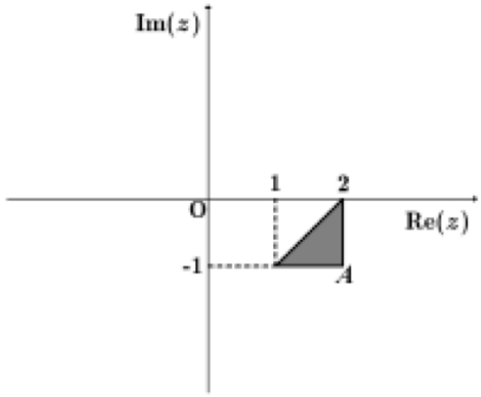
a) 1

b) -1

c) 2

d) None of the previous options.

11 (1/20) In the figure below, the shaded region represents some triangle on the complex plane. The point A has coordinates $(2, -1)$.



Which of the following conditions defines in $\mathbb{C}$ the shaded region, including its boundary?

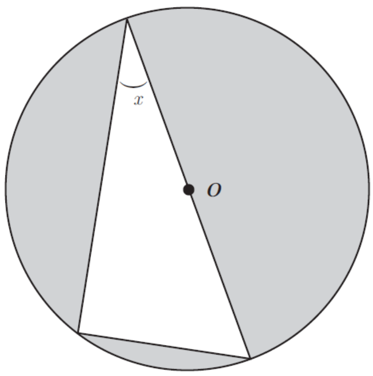
a) $|z-1| \geq |z-2+i| \ \wedge \ \mathrm{Re}(z) \leq 2 \ \wedge \ \mathrm{Im}(z) \geq -1$

b) $|z-1| \leq |z-2+i| \ \wedge \ \mathrm{Re}(z) \leq 2 \ \wedge \ \mathrm{Im}(z) \geq -1$

c) $|z-1| \geq |z-2+i| \ \wedge \ \mathrm{Re}(z) \geq -1 \ \wedge \ \mathrm{Im}(z) \leq 2$

d) None of the previous options.

12 (1/20) The figure below represents some triangle inscribed in a circle with center O and radius 1. One of the sides of the triangle is a diameter of the circle.



Which of the following expressions represents the area of the shaded region, as a function of x?

a) $\frac{\pi}{2} - \mathrm{sen}(2x)$

b) $\pi - \mathrm{sen}(2x)$

c) $\pi - 2\,\mathrm{sen}(2x)$

d) None of the previous options.

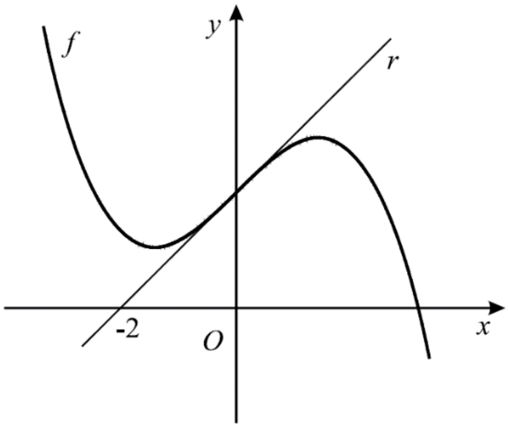
Group II

1 (3/20) Find the domain of the function f, of a real variable, defined by

$$f(x) = \frac{\log_{0.5}\left(x^2+x-2\right)}{\sqrt{e^{-4x} \cdot \left(\log_{10}\left(\left(\frac{1}{2}\right)^{x-7}\right)\right)^3 \cdot \left|\ln\left|\frac{x-2}{x^2-4}\right|\right|}},$$

justifying your reasoning and presenting all your computations. Present the result as a union of intervals.

2 (3/20) The figure below shows, on an orthonormal coordinate system xOy, part of the graph of a function f, with domain $\mathbb{R}$.



The straight line r, tangent to the graph of f at the point with abscissa 0, is parallel to the straight line bisecting the first and third quadrants and intercepts the Ox axis at the point with abscissa -2. Let $g: \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$g(x) = e^{xf(x)}.(x^3+2) + (f(x))^3 + \ln(\text{sen}^2 x + \cos^2 x) + \ln(\text{x}^4 + e) - \frac{2f(x)}{\sqrt[3]{\text{sen}^2 x + 1}}$$

for all $x \in \mathbb{R}$. Find the value of $g'(0)$, presenting all your computations.

3 (2/20) João wants to build a rectangular box with a square base and volume equal to 32 m^3, spending as little as possible with the acquisition of the necessary cardboard. Each square meter needed for the lateral parts, base and top costs, respectively, 1€, 3€ and 5€. Find the optimal dimensions of the box.

THE END

Solutions

Group I

1. c)
2. a)
3. b)
4. b)
5. a)
6. c)
7. a)
8. d)
9. c)
10. c)
11. a)
12. b)

Group II

1. $]-\infty, -3[\cup]-3, -2[\cup]1,2[\cup]2,7[$
2. 14
3. base side length: 2m; height: 8m.

Formulae

Progressions

Sum of the first n terms of a progression (u_n):

Arithmetic progression: $\frac{u_1+u_n}{2} \times n$

Geometric progression: $u_1 \times \frac{1-r^n}{1-r}$

Geometry

Area of the disk: πr^2

Area of the triangle: base $\times$ height $\times \frac{1}{2}$

Derivative Rules

$$(u+v)' = u' + v'$$
$$(u\,v)' = u'\,v + u\,v'$$
$$\left(\frac{u}{v}\right)' = \frac{u'\,v - u\,v'}{v^2}$$
$$(u^n)' = n\,u^{n-1}\,u' \quad (n \in \mathbb{R})$$
$$(\text{sen}\,u)' = u'\cos u$$
$$(\cos u)' = -u'\,\text{sen}\,u$$
$$(\text{tg}\,u)' = \frac{u'}{\cos^2 u}$$
$$(e^u)' = u'\,e^u$$
$$(a^u)' = u'\,a^u \ln a \quad (a \in \mathbb{R}^+ \setminus \{1\})$$
$$(\ln u)' = \frac{u'}{u}$$
$$(\log_a u)' = \frac{u'}{u \ln a} \quad (a \in \mathbb{R}^+ \setminus \{1\})$$

Special Limits

$$\lim\left(1+\frac{1}{n}\right)^n = e \quad (n \in \mathbb{N})$$