Mathematics Written Exam

Date	Duration + Tolerance	Elements of consultation allowed
26 May 2023	2 hours + 15 minutes	Calculator

Instructions
This exam is composed of two compulsory groups. Group I is composed of multiple choice questions and Group II is composed of open-ended questions. For each multiple choice question, four answer options are presented, of which exactly one is correct. Indicate your answer by selecting the letter corresponding to the option that you consider to be correct. Each correct, incorrect or blank/null answer in Group I has the classification of, respectively, 1/20, −0.33/20 and 0/20 points. In case more than one option is selected for the same question, the answer will be considered null. A negative total for this set of questions adds zero to the final grade. All open-ended questions should be carefully justified, and all computations should be presented. The classification for each question in this group is indicated next to it. Do not unstaple this booklet. Do not use any type of corrector. If necessary, cross out.

Name of the Candidate

Group I

1 (1/20) A student has to take two exams in the same day. The probability of getting a positive grade on the first exam is 0.6 and the probability of getting a negative grade on both exams is 0.2. What is the probability of the student getting a negative grade on the second exam, given that he has got a negative grade on the first exam?

a) 1/8

b) 1/3

c) 1/2

d) None of the above.

2 (1/20) Consider all the five digit numbers composed of numbers in $\{5, 6, 7, 8, 9\}$. What is the proportion of these numbers with exactly three 5 numbers in its composition?

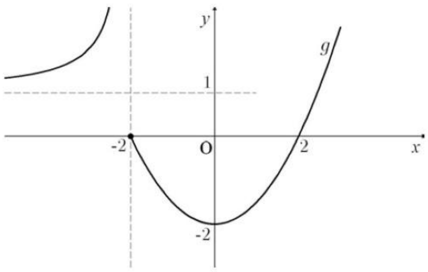
a) $(4/5)^3 \times 1/10$

b) $(4/9)^2 \times 1/7$

c) $(4/7)^2 \times 1/5$

d) None of the above.

3 (1/20) The figure below shows a portion of the graph of a function g, with domain $\mathbb{R}$ and continuous on $\mathbb{R} \setminus \{-2\}$. The straight lines with equations $x = -2$ and $y = 1$ are the only asymptotes of the graph of g.



Let (u_n) be a sequence such that $\lim_{n \to +\infty} g(u_n) = +\infty$. Which of the following expressions may be the general term of (u_n)?

a) $-2 + \frac{2}{n}$

b) $-2 - \frac{1}{n}$

c) $1 + \frac{1}{n}$

d) None of the above.

4 (1/20) Consider the sequence with general term $u_n = 2^{-n}$. What is the value of

$$\lim_{n\to+\infty}(u_1 + u_2 + \cdots + u_n)?$$

a) 0

b) 1

c) $+\infty$

d) None of the above.

5 (1/20) Consider an odd function f, with domain $\mathbb{R}$, such that $\lim_{x\to+\infty}(f(x) - 3x) = -1$. What is the value of $\lim_{x\to-\infty} f(x)$?

a) $+\infty$

b) -3

c) $-\infty$

d) None of the above.

6 (1/20) Consider the sequence (u_n) with general term

$$u_n = \begin{cases} (-1)^{2n}.\ln\left(\frac{4n+en^2}{n^2-1}\right), & \text{if } n \text{ even} \\ \frac{2^n+1}{2^n-3^n}.\left(\sqrt{e^{-n}} - 1\right), & \text{if } n \text{ odd} \end{cases}.$$

Which of the following propositions is true?

a) $\lim u_n = 0$

b) $\lim u_n = 1$

c) The sequence (u_n) does not admit limit.

d) None of the above.

7 (1/20) Consider, on an orthonormal referential in space, the points

$$A = (0,0,1), \qquad B = (5,0,2), \qquad C = (3,1,-3).$$

Which of the following propositions is true?

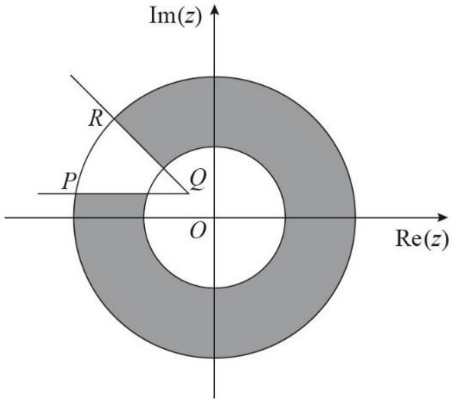
a) The vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ are perpendicular.

b) The points A, B and C are collinear.

c) The angle between vectors $\overrightarrow{AC}$ and $\overrightarrow{BC}$ is an acute angle.

d) None of the above.

8 (1/20) In the figure below, the shaded region represents a circular crown on the complex plane.



Consider that:

- O is the origin of the referential
- the point Q is the geometrical image of the complex number $-1+i$
- the straight line PQ is parallel to the real axis
- both circles are centered at the origin
- the radii of the circles are 3 and 6, respectively.

Given a complex number z, consider $\arg(z)$ to be the argument of z that belongs to the interval $[-\pi, \pi[$.

Which of the following conditions define, in $\mathbb{C}$, the shaded region, including its boundary?

a) $3 \leq |z| \leq 6 \ \wedge \ -\pi \leq \arg(z-1+i) \leq \frac{2\pi}{3}$

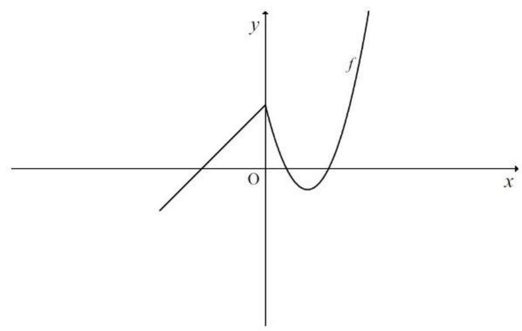
b) $9 \leq |z| \leq 36 \ \wedge \ -\pi \leq \arg(z+1-i) \leq \frac{3\pi}{4}$

c) $3 \leq |z| \leq 6 \ \wedge \ -\pi \leq \arg(z+1-i) \leq \frac{3\pi}{4}$

d) None of the above.

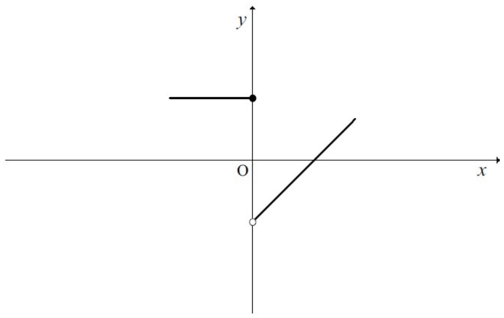
Group II

1 (1/20) The figure below shows a portion of the graph of a function f, with domain $\mathbb{R}$.

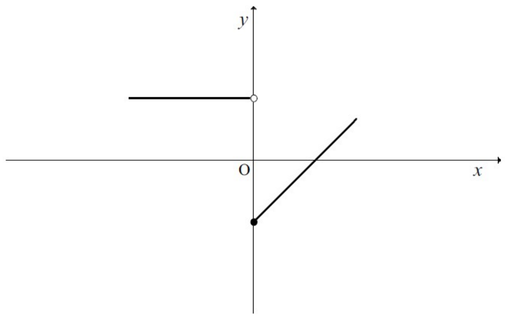


In which of the following figures may a portion of the graph of the derivative of f be found? <u>Justify conveniently.</u>

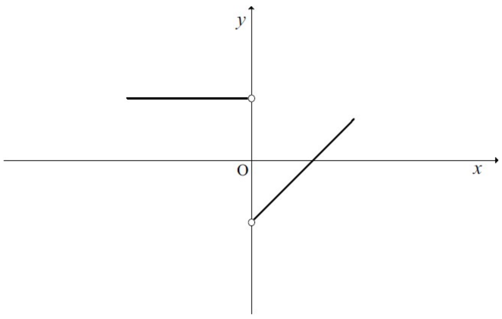
(A)



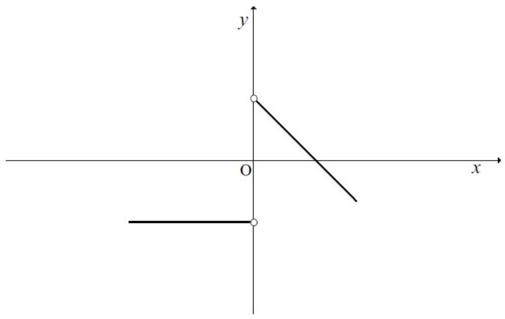
(B)



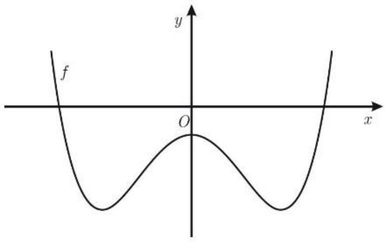
(C)



(D)



2 (2/20) The figure below shows a portion of the graph of a function f, with domain $\mathbb{R}$.



In which of the following expressions may $f''(x)$, for x in $\mathbb{R}$, be found? Justify conveniently.

a) $9 - x^2$

b) $(x+3)^2$

c) $-(x-3)^2$

d) $x^2 - 9$

3 (2/20) What is the domain of the function f, of a real variable, defined by

$$f(x) = \frac{\ln(|x|-1)\,\sqrt{9-x^2}}{\sqrt[3]{e^{x+1}\,.\,\ln(x)\,.\,\left(e^{x^2}-\sqrt{e}\right)}},$$

for all x? Present all your computations.

4 (2/20) Using exclusively analytical methods, find the set of all $x \in \mathbb{R}$ that satisfy

$$2^{-(x^2-1)\ln|x^2-2x|} < 1,$$

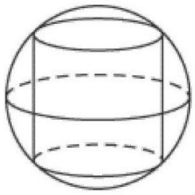
justifying your reasoning and presenting all your computations. Present the solution set as a union of intervals.

5 (2/20) Consider the straight line r, defined by $-2x = 1 - y$ on an orthonormal referential Oxy. Suppose that r is tangent to the graph of $f\colon \mathbb{R} \longrightarrow \mathbb{R}$, at the point with abscissa 0, and that $g\colon \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$g(x) = \sqrt[3]{f(x)\,.\,(2x+1)^2} - \frac{3x}{e^{\cos(2x-\pi)}} + \ln\left(\sqrt{x^2+1}\right),$$

for all $x \in \mathbb{R}$. Find the value of $g'(0)$, presenting all your computations.

6 The figure below shows a circular cylinder inscribed in a sphere of radius R.



a) (1/20) Show that r, the radius of the base of the cylinder, and h, the height of the cylinder, satisfy the relation

$$\left(\frac{h}{2}\right)^2 + r^2 = R^2.$$

b) (2/20) Find, as functions of R, the dimensions of the cylinder that maximize its volume.

THE END

Formulae

Progressions

Sum of the first n terms of a progression (u_n):

Arithmetic progression: $\frac{u_1+u_n}{2} \times n$

Geometric progression: $u_1 \times \frac{1-r^n}{1-r}$

Geometry

Area of the disk: πr^2

Volume of the cylinder: area of the base × height

Derivative Rules

$$(u+v)' = u' + v'$$

$$(u\,v)' = u'\,v + u\,v'$$

$$\left(\frac{u}{v}\right)' = \frac{u'\,v - u\,v'}{v^2}$$

$$(u^n)' = n\,u^{n-1}\,u' \quad (n \in \mathbb{R})$$

$$(\operatorname{sen} u)' = u' \cos u$$

$$(\cos u)' = -u' \operatorname{sen} u$$

$$(\operatorname{tg} u)' = \frac{u'}{\cos^2 u}$$

$$(e^u)' = u'\,e^u$$

$$(a^u)' = u'\,a^u \ln a \quad (a \in \mathbb{R}^+ \setminus \{1\})$$

$$(\ln u)' = \frac{u'}{u}$$

$$(\log_a u)' = \frac{u'}{u \ln a} \quad (a \in \mathbb{R}^+ \setminus \{1\})$$

Special Limits

$$\lim\left(1+\frac{1}{n}\right)^n = e \quad (n \in \mathbb{N})$$

Solutions

Group I

1. c)
2. a)
3. b)
4. b)
5. c)
6. c)
7. c)
8. c)

Group II

1. c)
2. d)
3. $]1,3]$
4. $]-\infty,-1[\cup]1-\sqrt{2},1[\cup]1+\sqrt{2},+\infty[$
5. $2-3e$

6.b) radius of the base of the cylinder: $\frac{\sqrt{6}}{3}R$; height of the cylinder: $\frac{2\sqrt{3}}{3}R$.